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High-Order Numerical Method for Two-Point Boundary Value Problems

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In this paper the two-point boundary value problem is transformed into general first-order ordinary differential equation system through introduction of conditions of an integral character to supplement the simultaneous set of first-order equations. A new discrete approximation of a high-order compact difference scheme is presented for the first-order system. It is a block-bidiagonal profile and removes the limits of other high-order discrete schemes at the interval ends. The numerical tests of a seventh-order compact difference scheme show that the proposed scheme is very convenient and efficient for linear and nonlinear two-point boundary value problems. © 1995 Academic Press, Inc.

1. INTRODUCTION

Until now, one can use many numerical methods to solve two-point boundary value problems. Generally speaking, these may be divided into initial value and boundary value methods. The initial value methods have some advantages: step sizes are adjusted flexibly, errors are controlled at each time step, little storage is needed, and so on. But they have a critical defect: it is difficult or impossible for some two-point boundary value problems to be solved with these methods. The reason is that even for very well-conditioned boundary value problems the corresponding initial value problems can be very ill-conditioned. In addition, using initial value methods in boundary value problems needs iteration procedures, so more computing time is spent. Although boundary value methods do not have some advantages of initial value methods, they can overcome the shortcomings of initial value methods and can be suited for different kinds of problems.

In the domains of plasma physics, hydrodynamics, and aerodynamics, two-point boundary value problems that are solved with difficulty by common numerical methods often appear, for instance, the singular perturbation case, the acute oscillatory case, the severe exponential case, and the intrinsic instability case. Employing the common initial value or low-order boundary methods will give inaccurate numerical results. Of course, one expects that there exist widely applicable and very efficient numerical methods to solve this kind of boundary problem. Usually there are two ways: adopting non-uniform meshes and using high-order discrete approximation schemes. Because the selection of non-uniform meshes is relative to concrete problems [5, 6], this way has no generalization. Quartapelle and Rebay [1] employed linear multipoint schemes to approximate the first-order equation system that is transformed from the two-point boundary value problem. They got fourth-order and sixth-order (actually fourth-order overall) bordered quadridiagonal schemes. To improve the accuracy of numerical solutions, they used Wilkinson's iterative refinement procedure. The highest overall accuracy of their approximation scheme is only fourth order because the scheme has limits of the accuracy at interval ends. The author presents a new method with which arbitrary order block-bidiagonal compact scheme can be constructed. The proposed scheme has no limits or the accuracy order of approximation scheme at the interval ends and can easily deal with all kinds of boundary conditions as well. In this paper, a seventh-order compact difference scheme is constructed and Newton–Cotes integration formula ($n = 6$) is used for conditions of integral character. Comparing the numerical results of the seventh-order scheme with the linear multipoint schemes in Ref. [1], we discover that the numerical results of the seventh-order scheme are very accurate.

The content of this paper is organized as follows. Section 2 describes the transformation of a two-point boundary value problem into a system of the first-order equations and the treatment of boundary conditions. In Section 3, a seventh-order bidiagonal compact difference scheme is constructed. Details are omitted. The computing process of the scheme is given in Appendix A with mid-variables to help readers edit the program conveniently. Section 4 shows some numerical results of the seventh-order scheme and linear multipoint schemes. Section 5 gives conclusions about the proposed scheme.

2. SYSTEM OF THE FIRST-ORDER TWO-POINT BOUNDARY VALUE PROBLEM AND INTEGRAL CONDITIONS

The first-order linear system of a two-point boundary value problem is

$$\mathbf{Y}' = \mathbf{A}(x)\mathbf{Y} + \mathbf{R}(x), \quad x \in [a, b], \quad (2.1)$$

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$$\mathbf{B}_1\mathbf{Y}(a) + \mathbf{B}_2\mathbf{Y}(b) = \mathbf{D}, \tag{2.2}$$

where $\mathbf{A}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are $m \times m$ matrixes; $\mathbf{Y}$, $\mathbf{R}$, and $\mathbf{D}$ are m-vectors.

It is well known that arbitrary order linear ordinary differential equations can be transformed into the first-order system (2.1). So numerical methods based on the system (2.1) are of universal adaptation. The formula of integral conditions is

$$\int_a^b \Phi(x)\mathbf{Y}(x)\,dx = \mathbf{D}, \tag{2.3}$$

where $\Phi(x)$ is a known function matrix, which is decided by the Green identity and conditions of the original two-point boundary value problem. Setting $\Phi(x) = \delta(x - a)\mathbf{B}_1 + \delta(x - b)\mathbf{B}_2$, where $\delta(x)$ is the delta function, formula (2.3) is equivalent to formula (2.2). Boundary condition (2.2) is a particular case of the integral condition (2.3). To compute condition (2.3), the general formula is

$$\mathbf{B}_1\mathbf{Y}(x_1) + \mathbf{B}_2\mathbf{Y}(x_2) + \cdots + \mathbf{B}_N Y(x_N) = \mathbf{D}, \tag{2.4}$$

where $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_N$ are $m \times m$ matrixes, $\mathbf{D}$ is a m-vector, $x_1, x_2, \ldots, x_N$ are grids. Formula (2.4) is provided by a numerical integration formula.

One considers a fourth-order equation

$$y^{(4)} + p(x)y''' + q(x)y'' + u(x)y' + v(x)y = f(x), \quad x \in [a, b], \tag{2.5}$$

supplemented with the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \quad y'(a) = \alpha', \quad y'(b) = \beta', \tag{2.6}$$

where $p(x)$, $q(x)$, $u(x)$, $v(x)$, and $f(x)$ are known functions and α, β, α', β', are constants.

Superscript (4) denotes the fourth derivative. Letting $y_4' = y_3$, $y_3' = y_2$, and $y_2' = y_1$, one can get

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix},$$

$$\mathbf{A}(x) = \begin{bmatrix} -p(x) & -q(x) & -u(x) & -v(x) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{2.7}$$

$$\mathbf{R}(x) = \begin{bmatrix} f(x) \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

y_4 has boundary condition $y_4(a) = \alpha$ or $y_4(b) = \beta$ and y_3 has boundary condition $y_3(a) = a'$ or $y_3(b) = \beta'$. The integral condition of y_2 is

$$\int_a^b y_2\,dx = \beta' - \alpha'. \tag{2.8}$$

In the Green identity

$$\int_a^b (y_4'''\phi + \phi''' y_4)\,dx = (\phi y_4'' - \phi' y_4' + \phi'' y_4)|_a^b, \tag{2.9}$$

the function $\phi(x)$ is required to satisfy $\phi(a) = \phi(b) = 0$ because of the unknowns $y_4''(a)$ and $y_4''(b)$. Setting $\phi(x) = (x - a)(x - b)$, formula (2.9) equals

$$\begin{aligned}\int_a^b (x-a)(x-b)y_4'''\,dx &= \int_a^b (x-a)(x-b)y_1\,dx \\ &= (b-a)(\alpha' + \beta') + 2(\beta - \alpha),\end{aligned} \tag{2.10}$$

which is the integral condition of y_1.

Provided that formulae (2.8) and (2.10) have general expressions of numerical integration, respectively,

$$d_1 y_2(x)_1 + d_2 y_2(x_2) + \cdots + d_N y_2(x_N) = \beta' - \alpha', \tag{2.11}$$

$$\begin{aligned} e_1 y_1(x_1) + e_2 y_1(x_2) + \cdots + e_N y_1(x_N) \\ = (b-a)(\alpha' + \beta') + 2(\beta - \alpha),\end{aligned} \tag{2.12}$$

the general boundary condition can be

$$\mathbf{B}_1 = \begin{bmatrix} e_1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} e_2 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ldots,$$

$$\mathbf{B}_{N-1} = \begin{bmatrix} e_{N-1} & 0 & 0 & 0 \\ 0 & d_{N-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

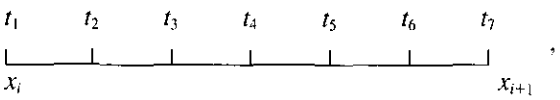
$$\mathbf{B}_N = \begin{bmatrix} e_N & 0 & 0 & 0 \\ 0 & d_N & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} (b-a)(\alpha' + \beta') + 2(\beta - \alpha) \\ \beta' - \alpha' \\ \beta' \\ \alpha \end{bmatrix}.$$

It is very convenient that the method to transform boundary conditions into general form (2.3) or (2.4) is used for separate, non-separate, and periodic conditions.

3. SEVENTH-ORDER COMPACT DIFFERENCE SCHEME

The idea of obtaining a high-order compact difference scheme is like this: First, transform a two-point boundary value problem into a general first-order equation system; then construct algebraic equations by means of the Taylor expansions of variables and their first-order derivatives as well as subjecting them to the first-order equation system at integral and fractional grids in each subinterval. Finally get a high-order block-bidiagonal compact difference scheme through some algebraic elimination processes with a computer. In this section, a seventh-order compact scheme is given with concrete procedures. Considering the discrete approximation of Eq. (2.1) in subinterval $[x_i, x_{i+1}]$, the subinterval is divided equally into six smaller subintervals as



where t_1, t_2, ..., t_7 are notations of grids in the subinterval $[x_i, x_{i+1}]$. Corresponding values of the variable and its derivative are marked as $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_7, \mathbf{Y}'_1, \mathbf{Y}'_2, ..., \mathbf{Y}'_7$. Taylor expansions of $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, $\mathbf{Y}_5$, $\mathbf{Y}_6$, and $\mathbf{Y}_7$ at the fractional grid t_4 can give six algebraic equations. Six unknown variables $\mathbf{Y}''_4, \mathbf{Y}'''_4, \mathbf{Y}_4^{(4)}, \mathbf{Y}_4^{(5)}, \mathbf{Y}_4^{(6)}$, and $\mathbf{Y}_4^{(7)}$ can be solved by a few algebraic operations, namely,

$$\frac{h^{n+1}}{(n+1)!}\mathbf{Y}_4^{(n+1)} = \sum_{j=1}^{7} a_j^n \mathbf{Y}_j + a_8^n h \mathbf{Y}'_4 + o(h^8 \mathbf{Y}_4^{(8)}), \quad (3.1)$$

where $h = (x_{i+1} - x_i)/6$, $n = 1, 2, ..., 6$; the coefficients a_j^n are shown in Appendix A.

Combining Taylor expansions of $\mathbf{Y}'_1, \mathbf{Y}'_2, \mathbf{Y}'_3, \mathbf{Y}'_5, \mathbf{Y}'_6, \mathbf{Y}'_7$ at grid t_4 with formulae (3.1) can give

$$Y'_k = \frac{1}{h}\sum_{j=1}^{7} b_j^k Y_j + b_8^k Y'_4 + o(h^7 Y_4^{(8)}), \quad k = 1, 2, 3, 5, 6, 7. \quad (3.2)$$

The coefficients b_j^k in formulae (3.2) are given in Appendix A.

Variable $\mathbf{Y}$ and its derivative $\mathbf{Y}'$ at grids t_1, t_2, ..., t_7 subject to equations

$$\mathbf{Y}'_j = \mathbf{A}_j \mathbf{Y}_j + \mathbf{R}_j, \quad j = 1, 2, ..., 7, \quad (3.3)$$

where $\mathbf{A}_j$ and $\mathbf{R}_j$ are values of $\mathbf{A}$ and $\mathbf{R}$ at grids t_j.

Substituting (3.3) into (3.2) can obtain six linear algebraic equations with respect to seven unknown variables, $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, $\mathbf{Y}_4$, $\mathbf{Y}_5$, $\mathbf{Y}_6$, and $\mathbf{Y}_7$. A formula of $\mathbf{Y}_1$ and $\mathbf{Y}_7$ can be gotten by eliminating $\mathbf{Y}_2$, $\mathbf{Y}_3$, $\mathbf{Y}_4$, $\mathbf{Y}_5$, and $\mathbf{Y}_6$ from the six equations, namely,

$$\frac{1}{h}\mathbf{S}_i \mathbf{Y}_i + \frac{1}{h}\mathbf{T}_i \mathbf{Y}_i = \mathbf{F}_i + o(h^7 \mathbf{Y}^{(8)}), \quad i = 1, 2, ..., N-1, \quad (3.4)$$

where $\mathbf{S}_i$ and $\mathbf{T}_i$ are $m \times m$ matrixes, $\mathbf{F}_i$ is a m-vector, given by computer. Formula (3.4) is a seventh-order compact difference scheme of Eq. (2.1) in the ith subinterval. The structure of the seventh-order scheme is profiled here as the following block-bordered bidiagonal matrix:

$$\begin{array}{cccccc} \mathbf{S}_1 & \mathbf{T}_1 & & & & \mathbf{F}_1 \\ & \mathbf{S}_2 & \mathbf{T}_2 & & & \mathbf{F}_2 \\ & & \ddots & \ddots & & \vdots \\ & & & \mathbf{S}_{N-1} & \mathbf{T}_{N-1} & \mathbf{F}_{N-1} \\ \mathbf{B}_1 & \mathbf{B}_2 & \cdots & \mathbf{B}_{N-1} & \mathbf{B}_N & \mathbf{D} \end{array}$$

Explicit procedures for $\mathbf{S}_i$, $\mathbf{T}_i$, and $\mathbf{F}_i$ are given in Appendix A. Adoption of selecting the row pivot in solving inverse matrixes and linear algebraic equations can avoid augmenting errors step by step. So the Wilkinson's refinement procedure is not needed. To match with the truncated errors of scheme (3.4), Newton–Cotes ($n = 6$) numerical integration formula,

$$\int_a^b f(x)\,dx = \frac{h}{140}(41f_1 + 216f_2 + 27f_3 + 272f_3 + 27f_5 + 216f_6 + 41f_7) - \frac{9h^9 f^{(8)}(\xi)}{1400}, \quad a < \xi < b, \quad (3.5)$$

is used in dealing with integral conditions. Therefore the number of grids should be $N = 6K + 1$; here K is an integral number.

4. NUMERICAL EXAMPLES AND COMPARISONS

In this section, several typical kinds of two-point boundary value problems are solved by the seventh-order compact difference scheme, whose solutions are found with difficulty by

TABLE I

Method	N	L^2 error	L^∞ error	Conver. order
	200	0.151(−1)	0.22(+1)	
Fourth-order	400	0.563(−1)	0.85(−1)	4.75
multipoint	800	0.337(−2)	0.51(−2)	4.06
scheme	1600	0.209(−3)	0.32(−3)	4.01
	3200	0.130(−4)	0.20(−4)	4.00
Seventh-order	103	0.289(−1)	0.410(−1)	
compact	199	0.148(−4)	0.209(−4)	11.42
scheme	397	0.426(−7)	0.602(−7)	8.44
	799	0.149(−9)	0.211(−9)	8.07

common numerical methods. The numerical results are compared with the multipoint schemes of Ref. [1].

4.1. *Oscillatory Case*

Considering the problem [1, 4]

$$y'' + \omega^2 y = S_0 \cos(\gamma x), \quad y(0) = 1, y(1) = 0,$$

the solution is

$$y(x) = C_1 \cos(\omega x) + C_2 \sin(\omega x) + S_0 \cos(\gamma x)/(\omega^2 - \gamma^2),$$

where $C_1 = 1 - S_0/(\omega^2 - \gamma^2)$, $C_2 = -[C_1 \cos(\omega) + S_0 \cos(\gamma)/(\omega^2 - \gamma^2)]/\sin(\omega)$. Setting $\omega^2 = 10^4$, $\gamma = 80$, and $S_0 = 10^4$, errors of the numerical solution and results of reference [1] are shown in Table I.

Setting $N = 1600$, errors of the numerical solution for several values of ω^2 are reported in Table II.

4.2. *Exponential Case*

An exponential example is the problem [1, 4]

$$y'' - \omega^2 y = S_0 \cos(\gamma x), \quad y(0) = 1, y(1) = 0,$$

whose solution is

TABLE II

ω^2	Sixth-order multipoint scheme L^2 error	Sixth-order multipoint scheme L^∞ error	Seventh-order compact scheme L^2 error	Seventh-order compact scheme L^∞ error
10^2	0.22(−11)	0.62(−11)	0.282(−12)	0.555(−12)
10^3	0.39(−9)	0.56(−9)	0.387(−12)	0.816(−12)
10^4	0.21(−6)	0.33(−6)	0.451(−11)	0.673(−11)
10^5	0.18(−3)	0.38(−3)	0.190(−8)	0.269(−8)

TABLE III

Method	N	L^2 error	L^∞ error	Conver. order
	200	0.221(−2)	0.557(−2)	
Fourth-order	400	0.123(−3)	0.291(−3)	4.17
multipoint scheme	800	0.744(−5)	0.170(−4)	4.05
	1600	0.461(−6)	0.130(−5)	4.01
Seventh-order	103	0.320(−4)	0.301(−3)	
compact scheme	199	0.456(−6)	0.501(−4)	6.41
	397	0.202(−8)	0.253(−7)	7.82
	799	0.749(−11)	0.996(−10)	7.99

$$y(x) = C_1 \exp(\omega x) + C_2 \exp(-\omega x) - (\omega^2 + \gamma^2)^{-1} S_0 \cos(\gamma x),$$

where $C_1 = (1 - e^{-2\omega})^{-1} e^{-\omega}[(\omega^2 + \gamma^2)^{-1} S_0(\cos(\gamma) - e^{-\omega}],$

$$C_2 = (1 - e^{-2\omega})^{-1}[1 + (\omega^2 + \gamma^2)^{-1} S_0(1 - e^{-\omega}\cos(\gamma)].$$

Selecting $\omega^2 = 10^4$, $\gamma = 80$, and $S_0 = 10^4$, errors of the numerical solution for the two schemes are given in Table III.

4.3. *Dichotomous Instability Case*

One can take the problem [1]

$$y'' - \omega^2 y = \omega^2 \cos^2(\pi x) + 2\pi^2 \cos(2\pi x), \quad y(0) = 0, y(1) = 0,$$

for example. The analytic solution is

$$y(x) = (1 + e^{-\omega})^{-1}[\exp(-\omega x) + \exp(\omega(x - 1))] - \cos^2(\pi x).$$

Giving $\omega = 20$, errors of the numerical solution for the two schemes are reported in Table IV.

Errors of the numerical solution with $\omega = 10^2$ are given in Table V by the seventh-order compact scheme.

4.4. *Turning Point Case*

A singular perturbation example with a turning point is the problem [1, 4]

TABLE IV

Method	N	L^2 error	L^∞ error	Conver. order
Fourth-order	100	0.17(−4)	0.38(−4)	
multipoint	200	0.67(−6)	0.18(−5)	4.67
scheme (with two	400	0.31(−7)	0.89(−7)	4.45
Wilkinson	800	0.17(−8)	0.47(−8)	4.18
refinement)	1600	0.10(−9)	0.27(−9)	4.05
	3200	0.64(−11)	0.15(−10)	4.01
Seventh-order	7	0.78(−5)	0.15(−4)	
compact scheme	13	0.45(−7)	0.11(−6)	7.44
	25	0.19(−9)	0.44(−9)	7.89
	97	0.39(−14)	0.93(−14)	7.79

TABLE V

N	L^2 error	L^∞ error	Conver. order
13	0.995(−3)	0.254(−2)	
25	0.169(−4)	0.599(−4)	5.88
103	0.312(−9)	0.167(−8)	7.53
199	0.154(−11)	0.804(−11)	8.01
403	0.629(−14)	0.349(−13)	7.77

$$\varepsilon y'' + xy' = -\varepsilon\pi^2\cos(\pi x) - \pi x\sin(\pi x),$$
$$y(-1) = -2, y(1) = 0,$$

whose solution is

$$y(x) = \cos(\pi x) + \frac{\operatorname{erf}(x/\sqrt{2\varepsilon})}{\operatorname{erf}(1/\sqrt{2\varepsilon})}.$$

Errors of the numerical solution with $\varepsilon = 10^{-4}$ are given in Table VI.

Making use of stretched interval transformation can obtain a better numerical solution for boundary layer problems [3].

4.5. *Nonlinear Equation with an End Singularity*

A nonlinear example with an end singularity is the problem [1, or 2, P36]

$$y'' + y'/x + \delta e^y = 0, \quad 0 < \delta \le 2, y'(0) = 0, y(1) = 0.$$

The solution is

$$y(x) = \ln\left[\frac{8C_\pm}{C_\pm x^2 + \delta^2}\right], \quad C_\pm = 4 - \delta \pm 2\sqrt{2(2-\delta)}.$$

Using l'Hopital's rule deals with the end singularity, i.e.,

$$2y'' + \delta e^y = 0 \quad \text{at } x = 0.$$

Starting from the initial guess $y(x) = 0$ and taking $\delta = 1$, the Newton method converges to the solution characterized by the root C_-. Errors of the numerical solution for the two schemes are shown in Table VII.

TABLE VI

Method	N	L^2 error	L^∞ error	Wilkinson iterations
Fourth-order multipoint scheme	600	0.212(−1)	0.282(−1)	2
	800	0.117(−1)	0.158(−1)	2
	1000	0.392(−2)	0.528(−2)	3
	1200	0.152(−2)	0.272(−2)	6
Seventh-order compact scheme	301	0.393(−2)	0.558(−2)	
	601	0.944(−6)	0.134(−5)	
	1201	0.113(−12)	0.838(−12)	

TABLE VII

Method	N	L^2 error	L^∞ error	Iterations
Fourth-order multipoint scheme	50	0.55(−7)	0.15(−6)	4
	100	0.41(−8)	0.12(−7)	4
	200	0.30(−9)	0.80(−9)	4
	400	0.28(−10)	0.60(−10)	4
Seventh-order compact scheme	7	0.26(−13)	0.36(−13)	4
	13	0.15(−15)	0.34(−15)	4

5. CONCLUSION

Through the numerical comparisons between the proposed scheme and the linear multipoint scheme, it is obvious that the seventh-order compact scheme is very powerful for two-point boundary value problems. The numerical errors of the proposed scheme are much smaller than ones of the linear multipoint schemes for the same number of grids (see the tables in Section 4). The deficiency of the proposed scheme is that expressions of the scheme are not apparent. One must edit the subroutine to obtain the scheme. Although constructing the scheme consumes more computer time than constructing a linear multipoint scheme, a small number of discrete grids are needed under a given tolerance error because of the high accuracy of the proposed scheme. In addition, the scheme is bidiagonal. Thus it takes less computer time to solve algebraic equations of the proposed scheme than to solve algebraic equations of a linear multipoint scheme at the same grids.

APPENDIX A

The coefficients a_j^n of formulas (3.1) are

$$a_1^1 = a_7^1 = \tfrac{1}{180}, \quad a_2^1 = a_6^1 = \tfrac{-3}{40}, \quad a_3^1 = a_5^1 = \tfrac{3}{4} \quad a_4^1 = \tfrac{-49}{36}, \quad a_8^1 = 0;$$
$$a_1^2 = -a_7^2 = \tfrac{-1}{540}, \quad a_2^2 = -a_6^2 = \tfrac{3}{80}, \quad a_3^2 = -a_5^2 = \tfrac{-3}{4}, \quad a_4^2 = 0, \quad a_8^2 = \tfrac{-49}{36};$$
$$a_1^3 = a_7^3 = \tfrac{-1}{144}, \quad a_2^3 = a_6^3 = \tfrac{1}{12}, \quad a_3^3 = a_5^3 = \tfrac{-13}{48}, \quad a_4^3 = \tfrac{7}{18} \quad a_8^3 = 0;$$
$$a_1^4 = -a_7^4 = \tfrac{1}{432}, \quad a_2^4 = -a_6^4 = \tfrac{-1}{24}, \quad a_3^4 = -a_5^4 = \tfrac{13}{48}, \quad a_4^4 = 0, \quad a_8^4 = \tfrac{7}{18};$$

$$a_1^5 = a_7^5 = \tfrac{1}{720}, \quad a_2^5 = a_6^5 = \tfrac{-1}{120}, \quad a_3^5 = a_5^5 = \tfrac{1}{48}, \quad a_4^5 = \tfrac{-1}{36}, \quad a_8^5 = 0;$$
$$a_1^6 = -a_7^6 = \tfrac{-1}{2160}, \quad a_2^6 = -a_6^6 = \tfrac{1}{240}, \quad a_3^6 = -a_5^6 = \tfrac{-1}{48}, \quad a_4^6 = 0, \quad a_8^6 = \tfrac{-1}{36}.$$

The coefficients b_j^k of formulas (3.2) are

$$b_j^1 = -6a_j^1 + 27a_j^2 - 108a_j^3 + 405a_j^4 - 1458a_j^5 + 5103a_j^6 + \text{sgn}(j-8),$$
$$b_j^2 = -4a_j^1 + 12a_j^2 - 32a_j^3 + 80a_j^4 - 192a_j^5 + 448a_j^6 + \text{sgn}(j-8),$$
$$b_j^3 = -2a_j^1 + 3a_j^2 - 4a_j^3 + 5a_j^4 - 6a_j^5 + 7a_j^6 + \text{sgn}(j-8),$$
$$b_j^5 = 2a_j^1 + 3a_j^2 + 4a_j^3 + 5a_j^4 + 6a_j^5\, 7a_j^6 + \text{sgn}(j-8),$$
$$b_j^6 = 4a_j^1 + 12a_j^2 + 32a_j^3 + 80a_j^4 + 192a_j^5 + 448a_j^6 + \text{sgn}(j-8),$$
$$b_j^7 = 6a_j^1 + 27a_j^2 + 108a_j^3 + 405a_j^4 + 1458a_j^5 + 5103a_j^6 + \text{sgn}(j-8),$$

where $j = 1, 2, ..., 8$; function sgn(x) is defined as

$$\text{sgn}(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Defining $\mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_{37}$ as $m \times m$ matrixes, $\mathbf{G}_1, \mathbf{G}_2, ..., \mathbf{G}_{10}$ as m-vectors, $\mathbf{I}$ as a $m \times m$ unit matrix, the computing procedure is executed in the following way:

$$c_1 = b_5^7 b_3^1 - b_5^1 b_3^7, \qquad c_2 = (b_6^7 b_5^1 - b_6^1 b_5^7)/c_1,$$
$$c_3 = (b_2^7 b_5^1 - b_2^1 b_5^7)/c_1, \qquad c_4 = (b_3^7 b_6^1 - b_3^1 b_6^7)/c_1,$$
$$c_5 = (b_3^7 b_2^1 - b_3^1 b_2^7)/c_1, \qquad c_6 = b_3^6 c_3 + b_5^6 c_5 + b_2^6,$$
$$c_7 = b_3^2 c_2 + b_5^2 c_4 + b_6^2; \qquad \mathbf{W}_1 = [(b_7^7 b_5^1 - b_7^1 b_5^7)\mathbf{I} - hb_5^1\mathbf{A}_7]/c_1,$$
$$\mathbf{W}_2 = [(b_1^7 b_5^1 - b_1^1 b_5^7)\mathbf{I} - hb_5^7\mathbf{A}_1]/c_1, \qquad \mathbf{W}_3 = [(b_4^7 b_5^1 - b_4^1 b_5^7)\mathbf{I} + h(b_8^7 b_5^1 - b_8^1 b_5^7)\mathbf{A}_4)]/c_1,$$
$$\mathbf{G}_1 = [b_5^7\mathbf{R}_1 - b_5^1\mathbf{R}_7 + (b_8^7 b_5^1 - b_8^1 b_5^7)\mathbf{R}_4)]/c_1; \qquad \mathbf{W}_4 = [(b_3^7 b_7^1 - b_3^1 b_7^7)\,\mathbf{I} + hb_3^1\mathbf{A}_7)]/c_1,$$
$$\mathbf{W}_5 = [(b_3^7 b_1^1 - b_3^1 b_1^7)\mathbf{I} - hb_3^7\mathbf{A}_1)]/c_1, \qquad \mathbf{W}_6 = [(b_3^7 b_4^1 - b_3^1 b_4^7)\mathbf{I} + h(b_3^7 b_8^1 - b_3^1 b_8^7)\mathbf{A}_4)]/c_1,$$
$$\mathbf{G}_2 = [b_3^1\mathbf{R}_7 - b_3^7\mathbf{R}_1 + (b_3^7 b_8^1 - b_3^1 b_8^7)\mathbf{R}_4)]/c_1; \qquad \mathbf{W}_7 = b_7^6\mathbf{I} + b_3^6\mathbf{W}_1 + b_5^6\mathbf{W}_4,$$
$$\mathbf{W}_8 = b_1^6\mathbf{I} + b_3^6\mathbf{W}_2 + b_5^6\mathbf{W}_5, \qquad \mathbf{W}_9 = b_4^6\mathbf{I} + b_3^6\mathbf{W}_3 + b_5^6\mathbf{W}_6 + hb_8^6\mathbf{A}_4,$$
$$\mathbf{W}_{10} = (b_3^6 c_2 + b_5^6 c_4 + b_6^6)\mathbf{I} - h\mathbf{A}_6, \qquad \mathbf{G}_3 = \mathbf{R}_6 - b_8^6\mathbf{R}_4 - b_3^6\mathbf{G}_1 - b_5^6\mathbf{G}_2;$$
$$\mathbf{W}_{11} = b_7^2\mathbf{I} + b_3^2\mathbf{W}_1 + b_5^2\mathbf{W}_4, \qquad \mathbf{W}_{12} = b_1^2\mathbf{I} + b_3^2\mathbf{W}_2 + b_5^2\mathbf{W}_5,$$
$$\mathbf{W}_{13} = b_4^2\mathbf{I} + b_3^2\mathbf{W}_3 + b_5^2\mathbf{W}_6 + hb_8^2\mathbf{A}_4, \qquad \mathbf{W}_{14} = (b_3^2 c_3 + b_5^2 c_5 + b_2^2)\mathbf{I} - h\mathbf{A}_2,$$
$$\mathbf{G}_4 = \mathbf{R}_2 - b_8^2\mathbf{R}_4 - b_3^2\mathbf{G}_1 - b_5^2\mathbf{G}_2; \qquad \mathbf{W}_{15} = b_7^5\mathbf{I} + b_3^5\mathbf{W}_1 + (b_5^5\mathbf{I} - h\mathbf{A}_5)\mathbf{W}_4,$$
$$\mathbf{W}_{16} = b_1^5\mathbf{I} + b_3^5\mathbf{W}_3 + (b_5^5 I - h\mathbf{A}_5)\mathbf{W}_5, \qquad \mathbf{W}_{17} = b_4^5\mathbf{I} + b_3^5\mathbf{W}_3 + (b_5^5\mathbf{I} - h\mathbf{A}_5)\mathbf{W}_6 + hb_8^5\mathbf{A}_4,$$
$$\mathbf{W}_{18} = (b_6^5 + b_3^5 c_2)\mathbf{I} + c_4\,(b_5^5\mathbf{I} - h\mathbf{A}_5), \qquad \mathbf{W}_{19} = (b_2^5 + b_3^5 c_3)\mathbf{I} + c_5(b_5^5\mathbf{I} - h\mathbf{A}_5),$$
$$\mathbf{G}_5 = \mathbf{R}_5 - b_8^5\mathbf{R}_4 - b_3^5\mathbf{G}_1 - (b_5^5\mathbf{I} - h\mathbf{A}_5)\mathbf{G}_2; \qquad \mathbf{W}_{20} = b_7^3\mathbf{I} + (b_3^3\mathbf{I} - h\mathbf{A}_3)\mathbf{W}_1 + b_5^3\mathbf{W}_4,$$
$$\mathbf{W}_{21} = b_1^3\mathbf{I} + (b_3^3\mathbf{I} - h\mathbf{A}_3)\mathbf{W}_2 + b_5^3\mathbf{W}_5, \qquad \mathbf{W}_{22} = b_4^3\mathbf{I} + (b_3^3\mathbf{I} - h\mathbf{A}_3)\mathbf{W}_3 + b_5^3\mathbf{W}_6 + hb_8^3\mathbf{A}_4,$$
$$\mathbf{W}_{23} = (b_5^3 c_4 + b_6^3)\mathbf{I} + c_2(\mathbf{b}_3^3\mathbf{I} - h\mathbf{A}_3), \qquad \mathbf{W}_{24} = (b_5^3 c_5 + b_2^3)\mathbf{I} + c_3\,(b_3^3\mathbf{I} - h\mathbf{A}_3),$$
$$\mathbf{G}_6 = \mathbf{R}_3 - b_8^3\mathbf{R}_4 - (b_3^3\mathbf{I} - h\mathbf{A}_3)\mathbf{G}_1 - b_5^3\mathbf{G}_2; \qquad \mathbf{W}_{28} = \mathbf{W}_{14}\mathbf{W}_{10} - c_6 c_7\mathbf{I},$$
$$\mathbf{W}_{25} = \mathbf{W}_{28}^{-1}(c_6\mathbf{W}_{11} - \mathbf{W}_{14}\mathbf{W}_7), \qquad \mathbf{W}_{26} = \mathbf{W}_{28}^{-1}(c_6\mathbf{W}_{12} - \mathbf{W}_{14}\mathbf{W}_8),$$
$$\mathbf{W}_{27} = \mathbf{W}_{28}^{-1}(c_6\mathbf{W}_{13} - \mathbf{W}_{14}\mathbf{W}_9), \qquad \mathbf{G}_7 = \mathbf{W}_{28}^{-1}(c_6\mathbf{G}_4 - \mathbf{W}_{14}\mathbf{G}_3);$$
$$\mathbf{W}_{29} = -(\mathbf{W}_{10}\mathbf{W}_{25} + \mathbf{W}_7)/c_6, \qquad \mathbf{W}_{30} = -(\mathbf{W}_{10}\mathbf{W}_{26} + \mathbf{W}_8)/c_6,$$
$$\mathbf{W}_{31} = -(\mathbf{W}_{10}\mathbf{W}_{27} + \mathbf{W}_9)/c_6, \qquad \mathbf{G}_8 = -(\mathbf{W}_{10}\mathbf{G}_7 + \mathbf{G}_3)/c_6;$$
$$\mathbf{W}_{32} = \mathbf{W}_{15} + \mathbf{W}_{18}\mathbf{W}_{25} + \mathbf{W}_{19}\mathbf{W}_{29}, \qquad \mathbf{W}_{33} = \mathbf{W}_{16} + \mathbf{W}_{18}\,\mathbf{W}_{26} + \mathbf{W}_{19}\mathbf{W}_{30},$$
$$\mathbf{W}_{34} = \mathbf{W}_{17} + \mathbf{W}_{18}\mathbf{W}_{27} + \mathbf{W}_{19}\mathbf{W}_{31}, \qquad \mathbf{G}_9 = \mathbf{G}_5 + \mathbf{W}_{18}\mathbf{G}_7 + \mathbf{W}_{19}\mathbf{G}_8;$$
$$\mathbf{W}_{35} = \mathbf{W}_{20} + \mathbf{W}_{23}\mathbf{W}_{25} + \mathbf{W}_{24}\mathbf{W}_{29}, \qquad \mathbf{W}_{36} = \mathbf{W}_{21} + \mathbf{W}_{23}\mathbf{W}_{26} + \mathbf{W}_{24}\mathbf{W}_{30},$$
$$\mathbf{W}_{37} = \mathbf{W}_{22} + \mathbf{W}_{23}\mathbf{W}_{27} + \mathbf{W}_{24}\mathbf{W}_{31}, \qquad G_{10} = \mathbf{G}_6 + \mathbf{W}_{23}\mathbf{G}_7 + \mathbf{W}_{24}\mathbf{G}_8;$$

$$\mathbf{S}_i = \mathbf{W}_{37}\mathbf{W}_{34}^{-1}\mathbf{W}_{33} - \mathbf{W}_{36}, \quad \mathbf{T}_i = \mathbf{W}_{37}\mathbf{W}_{34}^{-1}\mathbf{W}_{32} - \mathbf{W}_{35}, \quad \mathbf{F}_i = \mathbf{W}_{37}\mathbf{W}_{34}^{-1}\mathbf{G}_9 - \mathbf{G}_{10}.$$

Although writing directly the expressions of $\mathbf{S}_i$, $\mathbf{T}_i$, and $\mathbf{F}_i$ is tedious, it is convenient to do this task with a computer.

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